Two interacting Ising chains in relative motion

H.J. Hilhorst

Laboratoire de Physique Théorique, Bâtiment 210
Université Paris-Sud and CNRS, 91405 Orsay Cedex, France

February 1, 2011

Abstract

We consider two parallel cyclic Ising chains counter-rotating at a relative velocity v, the motion actually being a succession of discrete steps. There is an in-chain interaction between nearest-neighbor spins and a cross-chain interaction between instantaneously opposite spins. For velocities $v > 0$ the system, subject to a suitable markovian dynamics at a temperature T, can reach only a nonequilibrium steady state (NESS). This system was introduced by Hucht et al., who showed that for $v = \infty$ it undergoes a para- to ferromagnetic transition, essentially due to the fact that each chain exerts an effective field on the other one. The present study of the $v = \infty$ case determines the consequences of the fluctuations of this effective field when the system size N is finite. We show that whereas to leading order the system obeys detailed balancing with respect to an effective time-independent Hamiltonian, the higher order finite-size corrections violate detailed balancing. Expressions are given to various orders in N^{-1} for the interaction free energy between the chains, the spontaneous magnetization, the in-chain and cross-chain spin-spin correlations, and the spontaneous magnetization. It is shown how finite-size scaling functions may be derived explicitly. This study was motivated by recent work on a two-lane traffic problem in which a similar phase transition was found.

Keywords: kinetic Ising model, nonequilibrium stationary state, phase transition

LPT Orsay 11/03

1 Introduction

Recently Hucht [1] (see also [2]), motivated by the phenomenon of magnetic friction, formulated a nonequilibrium steady state (NESS) Ising model of a new type. It consists of two parallel linear Ising chains having a relative velocity v. In addition to a nearest-neighbor interaction in each chain, any pair of spins facing each other on the two chains has an instantaneous interaction. In the version of the model easiest to study, each chain is finite and periodic; we will therefore speak of cyclic counter-rotating Ising chains (CRIC). The model, subject to suitable temperature dependent Markovian dynamics, was shown [1] at velocity $v = \infty$ to have a para- to ferromagnetic phase transition which in the limit of infinitely long chains may be understood in terms of an equivalent equilibrium model.

The CRIC seems to us to be of the same fundamental importance as Glauber's [3] original kinetic Ising model. First, it is of interest in its own right as a new member of the class of NESS. Second, its interest is enhanced in the wider context of recent work on Ising models that in one way or another are driven, dissipate energy, or have some novel type of coupling; such work has appeared in a variety of contexts [4, 5, 6]. In particular, the present CRIC was extended to a Potts version by Iglói *et al.* [7], who find remarkable nonequilibrium phase transitions. In this paper we contribute further to the study of the CRIC. We focus on finite chains and on how to derive known and new properties from the master equation that defines the model.

Hucht's solution [1] is based on showing that at $v = \infty$ the stationary state dynamics of the CRIC is actually that of an equilibrium Ising chain in an effective magnetic field H_0, this field being zero above the transition temperature and nonzero below. This equivalence is valid in the limit where the chain length N tends to infinity. In this work we show that it is possible to formulate this problem as an expansion in powers of $N^{-1/2}$. To lowest order we recover the equivalent equilibrium system found in reference [1]. To higher orders fluctuations of the field H_0 come into play and appear as finite-size effects.

The finite N case is of interest, first of all, on the level of principles, and secondly, for the analysis of finite size effects in simulations as were carried out in [1] and by ourselves. We expect, furthermore, that our approach will help prepare the way for future work on the $v < \infty$ case, which is considerably harder.

The effective transition rates satisfy detailed balancing to leading order in the large-N expansion [1]; our analysis reveals, however, that to higher orders in $N^{-1/2}$ the detailed balancing (DB) symmetry of the effective rates is broken. The stationary state distribution may be found explicitly, at least

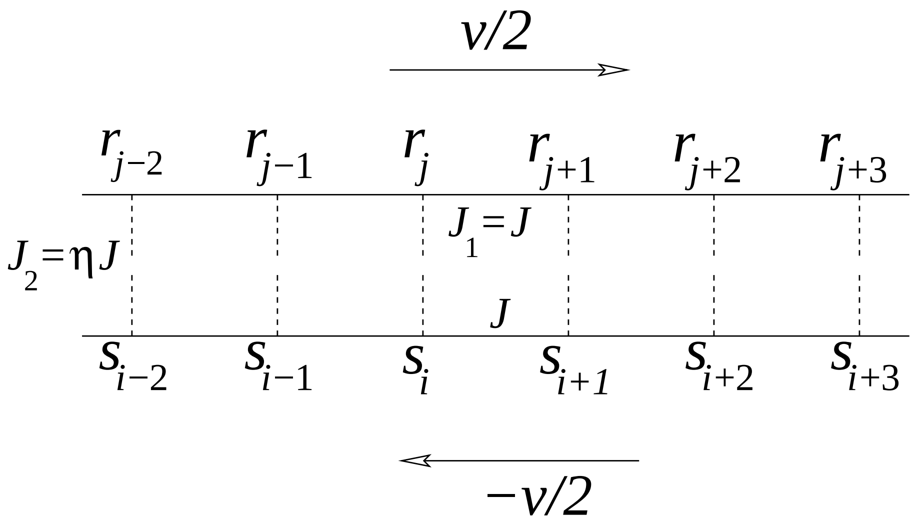

Figure 1: Ladder of spins with an intrachain nearest-neighbor interaction $J_1 = J$. The two chains constituting the ladder have a relative velocity v, the motion taking place in discrete steps of one lattice unit. There is an interchain nearest-neighbor interaction $J_2 = \eta J$ between each pair of spins facing each other at any instant in opposite chains.

to the lowest DB-violating order. Knowing this state one can calculate all desired NESS properties.

In section 2 of this paper we define the rules of the markovian dynamics for general relative velocity v and then specialize to $v = \infty$. These dynamical equations are the starting point for all that follows. In section 3 we discuss the DB violation that occurs in higher orders of N^{-1}. In section 4 we consider the stationary state to zeroth order, as was already done by Hucht [1]. In sections 5 and 6 we show how N^{-1} can be introduced as an expansion parameter and we define a 'leading order', composed of the zeroth order and a first-order correction. In section 7 we show how for the stationary state distribution an expansion may be found in powers of N^{-1}. We present the explicit result to next-to-leading order. In section 8 we calculate for various quantities of physical interest their stationary state averages to successive orders in the expansion. In section 9 we briefly discuss the relation of the present model to a two-lane road traffic model studied earlier. In section 10 we conclude.

2 Counter-rotating Ising chains

2.1 A stochastic dynamical system

We consider Ising spins on the ladder lattice shown in figure 1. The spins in the upper chain are denoted by r_j, those in the lower chain by s_i, where the integers j and i are site indices. There is a nearest-neighbor interaction $J_1 = J$ inside each chain and an interaction $J_2 = \eta J$ between each pair of spins facing each other in opposite chains. We take $J > 0$ and η of arbitrary sign. The feature [2] and [1] that distinguishes this model from the standard Ising model on a ladder lattice, is that the two chains move with respect to one another at a speed $v > 0$. This will mean the following: the time axis is discretized in intervals of duration $\tau = a_0/v$ (where a_0 is the lattice spacing) and at the end of each interval the upper chain is shifted one lattice spacing a_0 to the right with respect to the lower one. The Hamiltonian $\mathcal{H}(t)$ of this system is therefore time-dependent and given by

$$\mathcal{H}(t) = -J \sum_j \left[r_j r_{j+1} + s_j s_{j+1} \right] - \eta J \sum_j r_j s_{\lfloor j+vt/a_0 \rfloor} \,, \tag{2.1}$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to x.

We will consider cyclic boundary conditions[1]. In this case the chains become counter-rotating loops of length say N; the site indices i, j, and $\lfloor j + vt/a_0 \rfloor$ must then be interpreted modulo N. Employing the shorthand notation $r = \{r_j | j = 1, 2, \ldots, N\}$ and $s = \{s_j | j = 1, 2, \ldots, N\}$, we may indicate a spin configuration of the system by (r, s).

We associate with $\mathcal{H}(t)$ a stochastic time evolution of (r, s). Its precise definition requires that we exercise some caution. We will first define it as a Monte Carlo procedure and then write down the master equation and pass to analytic considerations. Single-spin reversals are attempted at uniformly distributed random instants of time at a rate of $1/\tau_0$ per site[2]. Each attempt is governed by transition probabilities. Since there are $2N$ sites, there are $2N$ different single-spin flips by which a state (r, s) may be entered or exited. Given that a reversal attempt takes place, let $(2N)^{-1} W_j^{\mathrm{r}}(r; s; t)$ and $(2N)^{-1} W_j^{\mathrm{s}}(s; r; t)$ be the probabilities that r_j and s_j are flipped, respectively. The reversal attempt will remain unsuccessful with the complementary probability

$$1 - A_{\mathrm{acc}} = 1 - (2N)^{-1} \sum_{j=1}^{N} \left[W_j^{\mathrm{r}}(r; s; t) + W_j^{\mathrm{s}}(s; r; t) \right] , \tag{2.2}$$

[1] In connection with the traffic problem open boundary conditions are certainly also worthy of consideration. These have however the inconvenience of breaking the translational symmetry.

[2] We may scale time such that $\tau_0 = 1$.

where A_{acc} is what is usually called the 'acceptance probability'.

We now specify the W_j^{r} and W_j^{r} in such a way that at any time t the system strives to attain the canonical equilibrium at a given temperature T with respect to the instantaneous Hamiltonian $\mathcal{H}(t)$. The choice is not unique. We choose

$$\begin{aligned} W_j^{\text{r}}(r;s;t) &= \tfrac{1}{4}\big[1-\tfrac{1}{2}r_j(r_{j-1}+r_{j+1})\tanh 2K\big]\big[1-r_js_i\tanh\eta K\big], \\ W_i^{\text{s}}(s;r;t) &= \tfrac{1}{4}\big[1-\tfrac{1}{2}s_i(s_{i-1}+s_{i+1})\tanh 2K\big]\big[1-s_ir_j\tanh\eta K\big], \end{aligned} \tag{2.3}$$

where we have set $K=J/T$ (with T measured in units of Boltzmann's constant) and where in *both* equations i and j are related by

$$i = \lfloor j + vt/a_0 \rfloor \mod N. \tag{2.4}$$

Equation (2.3) is different both from the heat bath (or: Glauber) and from the Metropolis transition probabilities. We will refer to it as the "factorizing rate". The factor

$$w_j^{\text{G}}(r) = \tfrac{1}{2}\big[1-\tfrac{1}{2}r_j(r_{j-1}+r_{j+1})\tanh 2K\big] \tag{2.5}$$

represents the Glauber transition probability. The W_j^{r} and W_j^{s} define an easy-to-simulate Markov chain[3] with time-dependent transition probabilities.[4]

In the special case $v=0$ the Hamiltonian $\mathcal{H}(t)$ reduces to the equilibrium Hamiltonian of the ladder lattice. For v arbitrary but $\eta=0$ it reduces to the equilibrium Hamiltonian of two decoupled chains. In both special cases the dynamics is standard and obeys detailed balancing.

In the general case, since the Hamiltonian is time-dependent, the system will not reach equilibrium but instead enter a NESS. Actually, for generic v, because of the periodic discrete shifts, the NESS is a τ-periodic function of time; NESS averages are naturally defined to include an average over this period. In the limiting case $v=\infty$ we have $\tau=0$ and this complication disappears. The infinite velocity NESS is the subject of our interest in the remaining sections. It is a problem that depends only on the two parameters K and η.

We note finally that as compared to ours, there is an extra prefactor

$$\frac{2+(1-r_{j-1}r_{j+1})\tanh 2K}{1+\tanh 2K}(1+\mathrm{e}^{-2\eta K}) \tag{2.6}$$

in Hucht's expression for the transition probability $W_j^{\text{r}}(r;s;t)$, and an analogous prefactor for $W_i^{\text{s}}(s;r;t)$. These factors may easily be carried along in the calculation.

[3]No confusion should arise with the two legs of the ladder lattice, to which we refer also as 'chains'.

[4]The reversal attempts, that is, the steps of the Markov chain, are Poisson distributed on the time axis. This makes it possible at any time to probabilistically connect the elapsed time t to the number of spin reversal attempts n. In the large t limit of course $n \simeq t/\tau_0$.

2.2 The limit $v \to \infty$

Let $P(r, s; n)$ be the probability distribution on the configurations (r, s) after n spin reversal attempts. We will now write down the formal evolution equation for $P(r, s; n)$ for the case of $v = \infty$, where important simplifications occur. When $v = \infty$ there is no relation between the indices i and j and hence the chain has transition probabilities $w_j(r; s)$ given by the average of (2.3) on all i, which is now considered as an independent variable. We denote this average by $w_j(r, s)$ and thus have

$$
\begin{aligned}
w_j(r; s) &= \frac{1}{N}\sum_{i=1}^{N} W_j^{\mathrm{r}}(r; s; t) \\
&= \tfrac{1}{4}\big[1 - \tfrac{1}{2} r_j (r_{j-1} + r_{j+1}) \tanh 2K\big]\big[1 - r_j \mu(s) \tanh \eta K\big] \\
&= w_j^{\mathrm{G}}(r) \times \tfrac{1}{2}\big[1 - r_j \mu(s) \tanh \eta K\big], \qquad (2.7\text{a}) \\
w_j(s; r) &= \frac{1}{N}\sum_{i=1}^{N} W_j^{\mathrm{s}}(s; r; t) \\
&= w_j^{\mathrm{G}}(s) \times \tfrac{1}{2}\big[1 - s_j \mu(r) \tanh \eta K\big], \qquad (2.7\text{b})
\end{aligned}
$$

where

$$\mu(s) = \frac{1}{N}\sum_{i=1}^{N} s_i, \qquad \mu(r) = \frac{1}{N}\sum_{i=1}^{N} r_i\,. \tag{2.8}$$

We will write r^j for the configuration obtained from r by reversing r_j (that is, by carrying out the replacement $r_j \mapsto -r_j$), and similarly define s^j. Summing on all $2N$ flips by which it is possible to enter or to exit (r, s) we find that the evolution of $P(r, s; n)$ is described by the master equation

$$
\begin{aligned}
P(r, s; n+1) = \frac{1}{2N}\sum_{j=1}^{N} \big[\, & w_j(r^j; s) P(r^j, s; n) + w_j(s^j; r) P(r, s^j; n) \\
& + \big(1 - w_j(r; s)\big) P(r, s; n) + \big(1 - w_j(s; r)\big) P(r, s; n) \,\big]\,,
\end{aligned} \tag{2.9}
$$

where the second line corresponds to the probability of an unsuccessful spin reversal attempt. In vector notation equation (2.9) may be written

$$P(n+1) = (\mathbf{1} + \mathcal{W}) P(n), \tag{2.10}$$

where $P(n)$ is the 2^{2N}-dimensional vector of elements $P(r, s; n)$, the symbol $\mathbf{1}$ denotes the unit matrix, and $\mathcal{W}$ is a matrix composed of entries w_j for

which comparison of (2.9) and (2.10) yields

$$\begin{aligned}\mathcal{W}(r,s;r',s') &= \delta_{r'r^j}\delta_{s's}w_j(r^j,s)+\delta_{r'r}\delta_{s's^j}w_j(r,s^j)\\ &\qquad -\delta_{r'r}\delta_{s's}\sum_{j=1}^{N}\left[w_j(r;s)+w_j(s;r)\right].\end{aligned} \tag{2.11}$$

The discrete-time master equation, (2.9) together with the Poisson statistics of the reversal attempts on the time axis, fully defines the CRIC for $v=\infty$. This equation may be studied analytically, as is the purpose of this work, or may be implemented in a Monte Carlo simulation.

3 Detailed balancing and its violation

Henceforth we consider the case $v=\infty$. Our purpose is now to find the stationary state distribution $P_{\rm st}(r,s)$ of the evolution equation (2.9). This distribution is the solution of $P(r,s;n)=P(r,s;n+1)=P_{\rm st}(r,s)$, which means

$$0=\mathcal{W}P_{\rm st}\,. \tag{3.1}$$

Combining equations (3.1) and (2.9) yields the $v=\infty$ stationary state equation

$$\begin{aligned}0 &= \sum_{j=1}^{N}\Big[w_j(r^j;s)P_{\rm st}(r^j,s)+w_j(s^j;r)P_{\rm st}(r,s^j)\\ &\qquad -w_j(r;s)P_{\rm st}(r,s)-w_j(s;r)P_{\rm st}(r,s)\Big].\end{aligned} \tag{3.2}$$

If the transition probabilities satisfy the condition of detailed balancing, the solution of (3.2) is easily constructed; in case of the contrary, there are no general methods. We examine therefore first the question of whether equation (2.9) satisfies detailed balancing.

A Markov chain satisfies detailed balancing (DB) if and only if its transition probabilities are such that any loop in configuration space is traversed with equal probability in either direction. To show that the transition probabilities w_j fail to obey DB we consider an elementary loop of four single-spin flips,

$$(r,s)\mapsto(r^j,s)\mapsto(r^j,s^j)\mapsto(r,s^j)\mapsto(r,s). \tag{3.3}$$

Given the system is in (r,s), we denote by $p_+(\eta)$ and $p_-(\eta)$ the probability that in the next four attempts it goes through this loop in forward and in backward direction, respectively. That is,

$$\begin{aligned}p_+(\eta) &= w_j(r;s)w_j(s;r^j)w_j(r^j;s^j)w_j(s^j;r),\\ p_-(\eta) &= w_j(s;r)w_j(r;s^j)w_j(s^j;r^j)w_j(r^j;s).\end{aligned} \tag{3.4}$$

For $\eta = 0$ the two chains are decoupled, and as discussed below equation (2.4), each of them separately satisfies DB; it is easy indeed to verify explicitly that $p_+(0) = p_-(0) \equiv p(0)$. For general η we may work out the difference $p_+(\eta) - p_-(\eta)$ with the aid of (2.7a), (2.8), and the relations

$$\mu(r^j) = \mu(r) - \frac{2r_j}{N}\,, \qquad \mu(s^j) = \mu(s) - \frac{2s_j}{N}\,, \tag{3.5}$$

which yields

$$\begin{aligned} p_+(\eta) - p_-(\eta) &= 4N^{-1}p(0)\,\tanh^2\eta K\,[r_j\mu(s) - s_j\mu(r)] \\ &\quad \times\big\{[r_j\mu(r) - s_j\mu(s)] + 2N^{-1}\tanh\eta K\big\}. \end{aligned} \tag{3.6}$$

This shows that DB is violated in the general case of nonzero coupling ($\eta \neq 0$) between the chains. It becomes valid again only asymptotically in the limit $N \to \infty$. We therefore cannot hope to rely on any general methods to construct $P_{\rm st}(r,s)$ for finite N. Indeed, writing out the stationary state equation (3.1) fully explicitly for $N = 3, 4$ (only $N = 2$ is trivial) has confirmed the nontriviality of the stationary state but has not provided us with any useful insight.

4 Stationary state $P_{\rm st}(r,s)$ to zeroth order

The limit $N \to \infty$ was considered by Hucht [1, 2], and we briefly recall the results. One may suppose that in this limit $\mu(r)$ and $\mu(s)$ have vanishing fluctuations around an as yet unknown common average to be called $m_0(K,\eta)$. We will denote the $N \to \infty$ limit of w_j by $w_{j,0}$. It then follows from (2.7a) that

$$w_{j,0}(r) = w_j^{\rm G}(r) \times \tfrac{1}{2}[1 - r_j m_0 \tanh\eta K]. \tag{4.1}$$

With the transition probabilities (4.1) the r- and the s-chain decouple. Moreover, the expression for these $w_{j,0}$ is such that the spin dynamics satisfies DB with respect to the pair of uncoupled nearest-neighbor Ising Hamiltonians in a field,

$$\mathcal{H}_0(r,s)/T = -K\sum_{j=1}^{N}\big[r_jr_{j+1} + s_js_{j+1}\big] - H_0\sum_{j=1}^{N}\big[r_j + s_j\big]\,. \tag{4.2}$$

where H_0 is defined in terms of m_0 by

$$\tanh H_0 = m_0 \tanh\eta K \tag{4.3}$$

and where K and H_0 both include a factor $1/T$. The quantity $\mathcal{H}_0(r,s)$ is an effective time-independent Hamiltonian. Let $m(K,z)$ denote the magnetization per spin of the one-dimensional (1D) Ising chain with coupling K in a

field that we will for convenience denote by z. This quantity is well-known and given by

$$m(K,z) = \frac{\sinh z}{\sqrt{\sinh^2 z + \mathrm{e}^{-4K}}}\,. \tag{4.4}$$

Consistency requires that

$$m_0 = m(K, H_0). \tag{4.5}$$

Upon combining (4.3) with (4.5) one obtains an equation for H_0 [or equivalently m_0]. The solution H_0 is a function of the two system parameters K and η and given by

$$\tanh H_0(K,\eta) = \begin{cases} \left(\dfrac{\tanh^2 \eta K - \mathrm{e}^{-4K}}{1 - \mathrm{e}^{-4K}} \right)^{\frac{1}{2}}, & K > K_{\mathrm{c}}, \\ 0, & K \leq K_{\mathrm{c}}\,, \end{cases} \tag{4.6}$$

in which there appears a critical coupling $K_{\mathrm{c}} = J/T_{\mathrm{c}}$ that is the solution of[5]

$$\tanh \eta K_{\mathrm{c}} = \mathrm{e}^{-2K_{\mathrm{c}}}. \tag{4.7}$$

The magnetization $m_0(K,\eta)$ follows directly from (4.3) and (4.6). For $T \to T_{\mathrm{c}}^-$ it vanishes as $m_0 \propto (T_{\mathrm{c}} - T)^{\beta}$ with a classical exponent $\beta = \frac{1}{2}$. For later use it is worthwhile to notice that also $H_0(T) \propto (T - T_{\mathrm{c}})^{1/2}$ when $T \leq T_{\mathrm{c}}$.

The DB property found below equation (4.1) now allows us to conclude that for $N \to \infty$ the stationary state distribution $P_{\mathrm{st},0}(r,s)$ is the Boltzmann distribution corresponding to (4.2), that is,

$$P_{\mathrm{st},0}(r,s) = \mathcal{N}_0\, \mathrm{e}^{-\mathcal{H}_0(r,s)/T} \tag{4.8}$$

where $\mathcal{N}_0$ is the normalization. In reference [1] several system properties were calculated in this $N \to \infty$ limit by averaging with respect to $P_{\mathrm{st},0}(r,s)$.

5 Expansion procedure for $P_{\mathrm{st}}(r,s)$

As has become clear in section 3, the inverse system size $1/N$ is a measure of the degree of DB violation. In the present case this will lead us to attempt to find the finite N stationary state by expanding around the known $N = \infty$ solution (4.8), which will play the role of the zeroth order result. At the basis

[5] Equation (4.7) may be rewritten as $\sinh(2J_1/T_{\mathrm{c}})\sinh(2J_2/T_{\mathrm{c}}) = 1$, which shows, as was also noticed in reference [1], that T_{c} is exactly (but accidentally) equal to the critical temperature of Onsager's square Ising model with horizontal and vertical couplings J_1 and J_2.

of the expansion is the hypothesis, to be confirmed self-consistently, that the fluctuations $\delta\mu$ of the chain magnetizations, defined by

$$\delta\mu(r) = \mu(r) - m_0\,, \qquad \delta\mu(s) = \mu(s) - m_0\,. \tag{5.1}$$

are of order $N^{-1/2}$.

A naive attempt to set up the expansion would be to notice that the transition probability (2.7a) can be written as a sum of its average and a correction, $w_j(r;s) = w_{j,0}(r) + \bar{w}_j(r;s)$, where $w_{j,0}(r)$ is given by (4.1) and $\bar{w}_j(r;s) = w_j^{\mathrm{G}}(r) \times (-\frac{1}{2}r_j)\delta\mu(s)\tanh\eta K$ is of order $N^{-1/2}$. One might then think that there exists a corresponding expansion $P_{\mathrm{st}}(r,s) = P_{\mathrm{st},0}(r,s)[1+\ldots]$. However, the dot terms turn out to be of order $\mathcal{O}(1)$ as $N\to\infty$, which is a sign that this is not the right way to expand. The reason for this failure is that P_{st} is the exponential of the extensive quantity $\mathcal{H}_0$; one should therefore ask first if this exponential contains any corrections of less divergent order in N before attempting to multiply it by a series of type $[1+\ldots]$. In the next section we describe how the expansion can be set up successfully.

Knowing how to calculate higher order corrections to the stationary state distribution, although certainly of diminishing practical interest, has a definite theoretical merit. What we will find in the end is that in fact to first order in the expansion detailed balancing continues to hold, but with respect to a Hamiltonian $\mathcal{H}^{(1)}(r,s)$ that acquires a first order correction. In section 6 we present the solution, to be denoted as $P_{\mathrm{st}}^{(1)}(r,s)$, of the stationary state to first order. In section 7 we will show how higher orders can be calculated and find that from the second order on DB violation appears. Section 7 also provides the demonstration of the correctness of the expansion.

6 Stationary state $P_{\mathrm{st}}(r,s)$ to first order

We use the upper index '(1)' to indicate any quantity correct up to first order in the expansion. We will prove that the correct expansion takes the form

$$P_{\mathrm{st}}(r,s) = P_{\mathrm{st}}^{(1)}(r,s)\,[1 + q_1(r,s) + q_2(r,s) + \ldots]\,, \tag{6.1}$$

where the q_k ($k=1,2,\ldots$), that we will show how to determine later, are of of order $\mathcal{O}(N^{-k/2})$ and where $P_{\mathrm{st}}^{(1)}(r,s)$, which includes a first order correction to the zeroth order result, is explicitly given by

$$P_{\mathrm{st}}^{(1)}(r,s) = \mathcal{N}^{(1)} \exp\left(-\frac{\mathcal{H}^{(1)}(r,s)}{T}\right), \tag{6.2a}$$

$$\frac{\mathcal{H}^{(1)}(r,s)}{T} = \frac{\mathcal{H}_0(r,s)}{T} - g_0 N \delta\mu(r)\delta\mu(s), \tag{6.2b}$$

$$g_0 = \cosh^2 H_0 \tanh \eta K, \tag{6.2c}$$

in which $\mathcal{N}^{(1)}$ is the appropriate normalization. The second term on the RHS of (6.2b) is a correction to the zeroth order effective Hamiltonian. It is $\mathcal{O}(1)$ for $N \to \infty$ and, since it is proportional to g_0, it vanishes as expected when $\eta = 0$.

In order to demonstrate (6.1)-(6.2) we split $\mathcal{W}$ according to

$$\mathcal{W} = \mathcal{W}^{(1)} + \sum_{k=2}^{\infty} \mathcal{W}_k \,, \tag{6.3}$$

where we take for $\mathcal{W}^{(1)}$ the matrix with the factorizing transition probabilities *that ensure detailed balancing with respect to* $\mathcal{H}^{(1)}$, and in which the $\mathcal{W}_k$ will be defined shortly. Expression (6.2) for Hamiltonian $\mathcal{H}^{(1)}$ shows that a spin r_j is subject to a total field $H_0 + g_0\delta\mu(s)$. Hence by analogy to (4.1) the transition probabilities that enter $\mathcal{W}^{(1)}$ are

$$w_j^{(1)}(r;s) = w_j^{\mathrm{G}}(r) \times \tfrac{1}{2}[1 - r_j \tanh\{H_0 + g_0\delta\mu(s)\}]. \tag{6.4}$$

We then have by construction that

$$\mathcal{W}^{(1)} P_{\mathrm{st}}^{(1)} = 0, \tag{6.5}$$

which is the combined zeroth and first order result. It may be obtained in explicit form from (3.2) by the substitutions $w_j \mapsto w_j^{(1)}$ and $P_{\mathrm{st}} \mapsto P_{\mathrm{st}}^{(1)}$.

A remark on terminology is in place at this point. Since the zeroth and first order will often be combined, we will refer to equation (6.5) as describing the 'leading order'. The terms q_1, q_2, $\ldots$ in the series (6.1) will be referred to as 'higher order' corrections.

7 Stationary state to higher orders

The validity of the expansion procedure of this section hinges on our being able to show that the corrections take effectively the form of the series of q_k in (6.1), where the terms are proportional to increasing powers of $N^{-1/2}$.

7.1 The perturbation series for $P_{\mathrm{st}}(r,s)$

In order to show that the higher order corrections to P_{st} can be expressed as the series of equation (6.1), we must first define the $\mathcal{W}_k$ in equation (6.3). Let us define $\delta w_j(r;s)$ by

$$w_j(r;s) = w_j^{(1)}(r;s) + \delta w_j(r;s). \tag{7.1}$$

Starting from (7.1) we employ the explicit expressions (2.7a) and (6.4) for w_j and $w_j^{(1)}$, respectively, perform a straightforward Taylor expansion in $\delta\mu$, and still use (4.3) to eliminate m_0 in favor of H_0. This leads to

$$\begin{aligned} \delta w_j(r;s) &= w_j^{\mathrm{G}}(r) \times \Big[\tfrac{1}{2}[1 - r_j\mu(s)\tanh\eta K] - \tfrac{1}{2}[1 - r_j\tanh\{H_0 + g_0\delta\mu(s)\}]\Big] \\ &= w_j^{\mathrm{G}}(r) \times (-\tfrac{1}{2}r_j)\Big[\{m_0 + \delta\mu(s)\}\tanh\eta K - \tanh\{H_0 + g_0\delta\mu(s)\}\Big] \\ &= w_j^{\mathrm{G}}(r) \times (-\tfrac{1}{2}r_j)\sum_{k=2}^{\infty} a_k\,\delta\mu^k(s) \\ &= \sum_{k=2}^{\infty} w_{j,k}(r;s), \end{aligned} \tag{7.2}$$

where the last equality, supposed to hold term by term in k, defines $w_{j,k}$ and shows that it is of order $N^{-k/2}$. In the third line of (7.2) the vanishing of the term linear in $\delta\mu$ has of course been pre-arranged. The first two coefficients a_k in that line are given by

$$\begin{aligned} a_2 &= g_0^2(1 - \tanh^2 H_0)\tanh H_0\,, \\ a_3 &= \tfrac{1}{3}g_0^3(1 - \tanh^2 H_0)(1 - 3\tanh^2 H_0). \end{aligned} \tag{7.3}$$

It becomes clear now that there is a qualitative difference between the high temperature regime $T \geq T_{\mathrm{c}}$ where we have $H_0 = 0, a_2 = 0$, and

$$a_3 = \frac{1}{3}\tanh^3\eta K, \qquad T \geq T_{\mathrm{c}}\,, \tag{7.4}$$

and the low temperature regime $T < T_{\mathrm{c}}$ where $H_0 > 0, a_2 > 0$.

We define the matrices $\mathcal{W}_k$ in expansion (6.3) in terms of the $w_{j,k}$ by analogy to (2.11). Hence for $T \geq T_{\mathrm{c}}$ we have that $\mathcal{W}_2 = 0$.

7.2 The higher order equations

The leading order equation (6.5) being satisfied, we now turn to the higher orders. Substitution of (6.3) in (3.1) and use of (6.5) leads to an expansion of which the first term is

$$\mathcal{W}^{(1)}P_{\mathrm{st}}^{(1)}q_1 + \mathcal{W}_2 P_{\mathrm{st}}^{(1)} = 0, \qquad T < T_{\mathrm{c}}\,. \tag{7.5a}$$

In the high temperature phase the fact that $\mathcal{W}_2 = 0$ implies that q_1=0 and therefore (7.5a) is replaced by the next term in the expansion,

$$\mathcal{W}^{(1)}P_{\mathrm{st}}^{(1)}q_2 + \mathcal{W}_3 P_{\mathrm{st}}^{(1)} = 0, \qquad T \geq T_{\mathrm{c}}\,. \tag{7.5b}$$

Either will be referred to as the 'next-to-leading order' equation. One obtains all higher-order equations in explicit form by inserting in the full stationary state equation (3.2) the expansions (6.1) for $P_{\mathrm{st}}(r,s)$ and (7.1)-(7.2) for $w_j(r;s)$.

7.3 Equation for $T < T_{\mathrm{c}}$

By the procedure indicated above we obtain for the next-to-leading order equation (7.5a) the explicit form

$$
\begin{aligned}
0 \;=\; \sum_j \Big[& w_{j,2}(r^j;s)P_{\mathrm{st}}^{(1)}(r^j,s) - w_{j,2}(r;s)P_{\mathrm{st}}^{(1)}(r,s) \\
& + w_{j,2}(s^j;r)P_{\mathrm{st}}^{(1)}(r,s^j) - w_{j,2}(s;r)P_{\mathrm{st}}^{(1)}(r,s) \\
& + w_j^{(1)}(r^j;s)P_{\mathrm{st}}^{(1)}(r^j,s)q_1(r^j,s) - w_j^{(1)}(r;s)P_{\mathrm{st}}^{(1)}(r,s)q_1(r,s) \\
& + w_j^{(1)}(s^j;r)P_{\mathrm{st}}^{(1)}(r,s^j)q_1(r,s^j) - w_j^{(1)}(s;r)P_{\mathrm{st}}^{(1)}(r,s)q_1(r,s)\Big].
\end{aligned}
\tag{7.6}
$$

We wish to divide (7.6) by $P_{\mathrm{st}}^{(1)}(r,s)$ and therefore have to compute

$$
\frac{P_{\mathrm{st}}^{(1)}(r^j,s)}{P_{\mathrm{st}}^{(1)}(r,s)} \equiv \mathrm{e}^{-2R_j(r;s)}. \tag{7.7}
$$

We easily find

$$
\begin{aligned}
2R_j(r;s) &= [\mathcal{H}^{(1)}(r^j,s) - \mathcal{H}^{(1)}(r,s)]/T \\
&= [\mathcal{H}_0(r^j,s) - \mathcal{H}_0(r,s)]/T - g_0 N\delta\mu(s)[\delta\mu(r^j) - \delta\mu(r)] \\
&= -2Kr_j(r_{j-1} + r_{j+1}) - 2r_j\{H_0 + g_0\delta\mu(s)\},
\end{aligned}
\tag{7.8}
$$

where we used (6.2) and (4.2). Detailed balancing says that

$$
w_j^{(1)}(r^j;s)P_{\mathrm{st}}^{(1)}(r^j,s)(r^j,s) = w_j^{(1)}(r;s)P_{\mathrm{st}}^{(1)}(r,s). \tag{7.9}
$$

Using (7.8) in the first two lines and (7.9) in the last two lines of (7.6) we obtain

$$
\begin{aligned}
0 \;=\; \sum_j \Big[& w_{j,2}(r^j;s)\mathrm{e}^{-2R_j(r;s)} - w_{j,2}(r;s) \\
& + w_{j,2}(s^j;r)\mathrm{e}^{-2R_j(s;r)} - w_{j,2}(s;r) \\
& + w_j^{(1)}(r;s)\{q_1(r^j,s) - q_1(r,s)\} \\
& + w_j^{(1)}(r;s)\{q_1(r,s^j) - q_1(r,s)\}\Big].
\end{aligned}
\tag{7.10}
$$

The expression in the first line of (7.10) may be rewritten as

$$
\begin{aligned}
& w_{j,2}(r^j;s)\mathrm{e}^{-2R_j(r;s)} - w_{j,2}(r;s) \\
&\quad = w_j^{\mathrm{G}}(r) \times \tfrac{1}{2}[1 - r_j \tanh H_0] \times 4r_j\delta\mu^2(s)g_0^2 \tanh H_0\,,
\end{aligned}
\tag{7.11}
$$

of which the first two factors on the RHS are again exactly $w_{j,0}$. In (7.10) $w_j^{(1)}$ is of order N^0 but contains corrections of higher order in $N^{-1/2}$. In (7.10), to leading order in $N^{-1/2}$, we may therefore replace it by its $N \to \infty$ limit, that is, by $w_{j,0}$ defined by (4.1). When we substitute (7.11) in (7.10) and apply to $w_j^{(1)}$ the $N \to \infty$ limit, we obtain the final form of the equations for the next-to-leading order correction to the stationary state,

$$\begin{aligned} 0 \;=\; \sum_j \Big[& w_{j,0}(r) \times 4r_j g_0^2 \tanh H_0\, \delta\mu^2(s) \\ & + w_{j,0}(s) \times 4s_j g_0^2 \tanh H_0\, \delta\mu^2(r) \\ & + w_{j,0}(r)\{q_1(r^j, s) - q_1(r,s)\} \\ & + w_{j,0}(s)\{q_1(r, s^j) - q_1(r,s)\}\Big]. \end{aligned} \tag{7.12}$$

7.4 Equation for $T \geq T_{\rm c}$

For $T \geq T_{\rm c}$ we have $a_2 = 0$ whence $q_1 = 0$. Equation (7.5b), when rendered explicit, leads to expressions that are identical to successively (7.6), (7.10), and (7.12) apart from the substitutions $q_1 \mapsto q_2$ and $w_{j,2} \mapsto w_{j,3}$. In this case $w_{j,0}(r) = \frac{1}{2} w_j^{\rm G}(r)$ where $w_j^{\rm G}(r)$ is given by (2.5), and $\delta\mu = \mu$ since $H_0 = m_0 = 0$. Hence instead of (7.12) we get

$$\begin{aligned} 0 \;=\; \sum_j \Big[& w_j^{\rm G}(r) \times 4r_j a_3 \mu^3(s) \\ & + w_j^{\rm G}(s) \times 4s_j a_3 \mu^3(r) \\ & + w_j^{\rm G}(r)\{q_2(r^j, s) - q_2(r,s)\} \\ & + w_j^{\rm G}(s)\{q_2(r, s^j) - q_2(r,s)\}\Big]. \end{aligned} \tag{7.13}$$

Finding the solutions of (7.12) and (7.13) will be the subject of the next two subsections. We will first consider the easier case of $T \geq T_{\rm c}$ and then the case $T < T_{\rm c}$.

7.5 Solution for $T \geq T_{\rm c}$

We start with the high temperature phase, where equation (7.13) applies. Detailed balancing would be satisfied if the expression under the sum on j were zero, that is, if we had

$$\begin{aligned} q_2(r^j, s) - q_2(r,s) &= -a_3 r_j \mu^3(s), \\ q_2(r, s^j) - q_2(r,s) &= -a_3 s_j \mu^3(r). \end{aligned} \tag{7.14}$$

It can easily be shown that it is impossible to satisfy these equations. However, they suggest that we look for a solution q_2 of the form

$$q_2(r,s) = NC_2 a_3[\mu(r)\mu^3(s) + \mu(s)\mu^3(r)] \tag{7.15}$$

where only the constant C_2 is still adjustable. The difference $q_2(r^j,s) - q_2(r,s)$ is easy to calculate, but we are interested only in its leading order. This leads to

$$\begin{aligned} q_2(r^j,s) - q_2(r,s) &= -2C_2 a_3 r_j[\mu^3(s) + 3\mu(s)\mu^2(r)] + \mathcal{O}(N^{-2}), && \text{(7.16a)} \\ q_2(r,s^j) - q_2(r,s) &= -2C_2 a_3 s_j[\mu^3(r) + 3\mu(r)\mu^2(s)] + \mathcal{O}(N^{-2}). && \text{(7.16b)} \end{aligned}$$

It should be noted that whereas (7.15) is of order N^{-1}, the differences (7.16) are of order $N^{-3/2}$. We now need

$$\sum_j w_j^{\mathrm{G}}(r)\{q_2(r^j,s) - q_2(r,s)\} = -2C_2 a_3 \left(\sum_j w_j^{\mathrm{G}}(r) r_j\right)[\mu^3(s) + 3\mu(s)\mu^2(r)]. \tag{7.17}$$

With the aid of the explicit expression for $w_j^{\mathrm{G}}(r)$ one evaluates easily

$$\sum_j w_j^{\mathrm{G}}(r) r_j = \tfrac{1}{4}(1-\gamma)N\mu(r). \tag{7.18}$$

we see that the equation is satisfied for $C_2 = \frac{1}{8}$. Hence from (7.15) we get

$$q_2(r,s) = \tfrac{1}{24}N(\tanh^3 \eta K)[\mu(r)\mu^3(s) + \mu(s)\mu^3(r)]. \tag{7.19}$$

This is of order N^{-1}.

7.6 Solution for $T < T_\mathrm{c}$

In the low-temperature regime equation (7.12) applies. In order to solve this equation we now postulate

$$q_1(r,s) = NC_1 b_2[\delta\mu^3(r) + \delta\mu^3(s)] \tag{7.20}$$

where C_1 is an adjustable constant and

$$b_2 = 4a_2/(1 - \tanh^2 H_0) = 4g_0^2 \tanh H_0\,. \tag{7.21}$$

Expression (7.20) is of order $N^{-1/2}$. Instead of (7.16) we now have the difference

$$q_1(r^j,s) - q_1(r,s) = -6C_1 b_2 r_j \delta\mu^2(r) + \mathcal{O}(N^{-3/2}). \tag{7.22}$$

which is of order N^{-1}. The first two lines of (7.12) require that we evaluate

$$\sum_j w_{j,0}(r)r_j = \tfrac{1}{4}\sum_j [1 - \tfrac{1}{2}\gamma r_j(r_{j-1} - r_{j+1})][1 - r_j \tanh H_0]r_j \qquad (7.23)$$

Unlike the sum in (7.17), this is not a sum of zero-average random terms. It will produce a result of order N, which we may replace by its average. This yields

$$\begin{aligned} \sum_j w_{j,0}(r)r_j &= \tfrac{1}{4}N[(1-\gamma)m_0 - (1-\gamma a_H)\tanh H_0] \\ &\equiv NG, \end{aligned} \qquad (7.24)$$

where the last equality defines G and where a_H is the nearest neighbor spin-spin correlation $\langle r_j r_{j+1}\rangle$ of a 1D Ising chain in a field as described by $\mathcal{H}_0$ [equation (4.2)]. Expression (7.24), contrary to its $T \geq T_{\mathrm{c}}$ counterpart (7.18), has no spin dependence and is therefore equal for the r- and s- spins. The first two lines of (7.12), to be denoted S_1, become

$$S_1 = 4NGg_0^2 \tanh H_0 \left[\delta\mu^2(s) + \delta\mu^2(r)\right]. \qquad (7.25)$$

We use (7.22) to write the last two lines of (7.12) as

$$\begin{aligned} S_2 &= -6C_1 b_2 \left[\Big(\sum_j w_{j,0}(r)r_j\Big)\delta\mu^2(r) + \Big(\sum_j w_{j,0}(s)s_j\Big)\delta\mu^2(s)\right] \\ &= -6NGC_1 b_2 \left[\delta\mu^2(r) + \delta\mu^2(s)\right]. \end{aligned} \qquad (7.26)$$

The stationary state equation (7.12) may the be written as $S_1 + S_2 = 0$ and we see that it is satisfied for $C_1 = \frac{1}{6}$.

$$q_1(r,s) = \tfrac{2}{3}Ng_0^2 \tanh H_0 \left[\delta\mu^3(r) + \delta\mu^3(s)\right]. \qquad (7.27)$$

7.7 Section summary

We have studied in the preceding subsections the large-N expansion of the stationary state distribution $P_{\mathrm{st}}(r,s)$ of the infinite velocity CRIC defined in section 2. We have shown, for $T < T_{\mathrm{c}}$ and $T \geq T_{\mathrm{c}}$ separately, the existence of a series of correction terms q_k that multiplies the leading order result $P_{\mathrm{st}}^{(1)}$ in (6.1), which itself is again composed of a zeroth and a first order contribution. This expansion also furnishes the necessary proof that the prefactor $P_{\mathrm{st}}^{(1)}$ represents indeed the 'leading order' behavior. We have determined explicitly the first nonzero correction term in this series: q_1 for $T \geq T_{\mathrm{c}}$ and q_2 for $T < T_{\mathrm{c}}$.

When looking ahead beyond this leading order correction, it appears that the q_k (for $k \geq 2$ when $T < T_{\rm c}$ and for $k \geq 3$ when $T \geq T_{\rm c}$) involve not only $\delta\mu(r)$ and $\delta\mu(s)$, but also energy fluctuations such as $N^{-1}\sum_j (r_j r_{j+1} - a_H)$, if not longer-range correlations. Therefore, even though on the basis of the results of this section one might be tempted to postulate a general solution of the simple type $P_{\rm st}(r,s) = P_{\rm st}^{(1)}(r,s)Q(\delta\mu(r), \delta\mu(s))$, it is unlikely that the true $P_{\rm st}(r,s)$ is of this form.

8 Stationary state averages

Stationary state averages of observables $A(r,s)$ are averages with respect to $P_{\rm st}(r,s)$, so that using (6.1) and (6.2a) we have

$$\begin{aligned} \langle A \rangle &= \frac{\sum_{r,s} A(r,s) \mathrm{e}^{-\mathcal{H}^{(1)}(r,s)/T}[1 + q_1(r,s) + q_2(r,s) + \ldots]}{\sum_{r,s} \mathrm{e}^{-\mathcal{H}^{(1)}(r,s)/T}[1 + q_1(r,s) + q_2(r,s) + \ldots]} \\ &= \langle A \rangle^{(1)} + [\langle A q_\ell \rangle^{(1)} - \langle A \rangle^{(1)} \langle q_\ell \rangle^{(1)}] + \ldots, \end{aligned} \tag{8.1}$$

where $\langle \ldots \rangle^{(1)}$ indicates an average with weight $P_{\rm st}^{(1)}(r,s)$ [equation (6.2)], the second line results from a straightforward expansion, and

$$\ell = \begin{cases} 2, & T \geq T_{\rm c}\,, \\ 1, & T < T_{\rm c}\,, \end{cases} \tag{8.2}$$

for the lowest order nonzero terms in the expansion. Although the q_k are accompanied by increasing powers of $N^{-1/2}$, the order in $N^{-1/2}$ of each of the terms in the series (8.1) must be analyzed for each observable A separately.

8.1 Integral representation of the partition function

The denominator in the first line of (8.1) is a normalization factor to which we may refer (although slightly improperly) as the partition function Z. In order to find expressions for the averages $\langle \ldots \rangle^{(1)}$ in the second line of (8.1), we begin by evaluating Z to leading order,

$$Z^{(1)}(K, H_0, g_0) \equiv \sum_{r,s} \mathrm{e}^{-\mathcal{H}^{(1)}(r,s)/T}, \tag{8.3}$$

with $\mathcal{H}^{(1)}$ given by (6.2b) in which one should substitute (4.2) and (5.1). To this order (8.3) is a true partition function, *viz.* the trace of a Boltzmann factor. The notation $Z^{(1)}(K, H_0, g_0)$ is meant to indicate that we wish to consider this quantity as a function of three independent parameters, ignoring

for the moment expression (6.2c) for g_0. The r- and s-spins in (8.3) may be decoupled by the integral representation

$$
\begin{aligned}
Z^{(1)} \;=\;& \frac{N}{\pi g_0}\int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}y\;\mathrm{e}^{-g_0^{-1}N(x^2+y^2)} \\
&\times\left[\mathrm{e}^{-(x+\mathrm{i}y)Nm_0}\sum_r \mathrm{e}^{K\sum_j r_j r_{j+1}+(H_0+x+\mathrm{i}y)\sum_j r_j}\right] \\
&\times\left[\mathrm{e}^{-(x-\mathrm{i}y)Nm_0}\sum_s \mathrm{e}^{K\sum_j s_j s_{j+1}+(H_0+x-\mathrm{i}y)\sum_j s_j}\right]
\end{aligned}
\tag{8.4}
$$

in which $m_0 = m(K, H_0)$ follows from (4.3) and (4.6). The two factors in brackets in (8.4) are seen to be the partition functions $\zeta(K, H_0 + x \pm \mathrm{i}y)$ of independent standard Ising chains in magnetic fields $H_0 + x \pm \mathrm{i}y$. Hence

$$
Z^{(1)} = \frac{N}{\pi g_0}\int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}y\,\mathrm{e}^{-g_0^{-1}N(x^2+y^2)-2xNm_0}\left|\zeta(K,H_0+x+\mathrm{i}y)\right|^2. \tag{8.5}
$$

We recall that

$$
\zeta(K,B)\equiv\lambda_+^N+\lambda_-^N\,, \tag{8.6}
$$

where

$$
\lambda_\pm(K,B)=\mathrm{e}^K\left[\cosh B\pm\sqrt{\sinh^2 B+\mathrm{e}^{-4K}}\right]. \tag{8.7}
$$

are the transfer matrix eigenvalues.

8.2 Stationary point and fluctuations

The x and y integrals in (8.5) are easily evaluated by the saddle point meyhod, In the limit of large N, we may neglect in (8.6) the exponentially small corrections due to λ_- and get from (8.5)

$$
Z^{(1)} \simeq \frac{N}{\pi g_0}\int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}y\;\mathrm{e}^{-N\mathcal{F}(x,y)}, \tag{8.8}
$$

where

$$
\mathcal{F}(x,y)=g_0^{-1}(x^2+y^2)+2xm_0-\log\left|\lambda_+(K,H_0+x+\mathrm{i}y)\right|^2 \tag{8.9}
$$

Let (x^*, y^*) denote the stationary point of the integration in (8.8). The stationary point equations $\mathcal{F}_x = \mathcal{F}_y = 0$ can be expressed as

$$
g_0^{-1}(x^*\pm\mathrm{i}y^*)=m(K,H_0+x^*\mp\mathrm{i}y^*)-m_0\,, \tag{8.10}
$$

with the magnetization $m(K,B) = \lambda_+^{-1}(K,B)\partial \log\lambda_+(K,B)/\partial B$ given by (4.4). For reasons of symmetry the stationary point must have $y^* = 0$. This reduces (8.10) to the single real equation

$$g_0^{-1}x^* = m(K, H_0 + x^*) - m(K, H_0), \tag{8.11}$$

where we used that $m_0 = m(K, H_0)$ [equation (4.5)]. Equation (8.11) has for all H_0 the obvious solution $x^* = 0$. We investigate the stability of the stationary point (x^*, y^*) by calculating the matrix of second derivatives,

$$\begin{aligned} &\mathcal{F}^*_{xx} = 2[g_0^{-1} - \chi(K,H_0)], \qquad \mathcal{F}^*_{yy} = 2[g_0^{-1} + \chi(K,H_0)], \\ &\mathcal{F}^*_{xy} = \mathcal{F}^*_{yx} = 0, \end{aligned} \tag{8.12}$$

where the asterisk indicates evaluation in the stationary point and where $\chi(K,B) = \partial m(K,B)/\partial B$ is the magnetic susceptibility. We obtain the eigenvalues $\mathcal{F}^*_{xx}$ and $\mathcal{F}^*_{yy}$ explicitly by substituting in (8.12) for g_0 the expressions (6.2c) and for χ the expression

$$\chi(K,B) = \frac{\mathrm{e}^{-4K}\cosh B}{\left(\sinh^2 B + \mathrm{e}^{-4K}\right)^{3/2}}\,, \tag{8.13}$$

where (4.4) has been used. This yields

$$\mathcal{F}^*_{xx,yy} = \begin{cases} \dfrac{2(\mathrm{e}^{-2K} \mp \tanh\eta K)}{\mathrm{e}^{-2K}\tanh\eta K}, & T > T_\mathrm{c}\,, \\[2ex] \dfrac{2(1-\tanh^2\eta K)(\tanh^2\eta K \mp \mathrm{e}^{-4K})}{(1-\mathrm{e}^{-4K})\tanh^3\eta K}, & T < T_\mathrm{c}\,, \end{cases} \tag{8.14}$$

in which the upper (lower) sign refers to the xx (to the yy) derivative. It can be seen that $\mathcal{F}^*_{yy}$ is positive for all temperatures, but that $\mathcal{F}^*_{xx}$, which is positive in both the high and the low-temperature phase, vanishes as $T \to T_\mathrm{c}$. Hence for all $T \neq T_\mathrm{c}$ the stability is ensured by the quadratic terms in the expansion of $\mathcal{F}(x,y)$ around the stationary point.

8.3 Free energy

We are now in a position to calculate various physical quantities of interest. The first one will be the *interaction* free energy per spin between the two chains which (divided by T) will be called F_int. It will turn out to have an expansion

$$F_\mathrm{int} = F_\mathrm{int}^{(0)} + N^{-1} f_\mathrm{int} + \ldots \tag{8.15}$$

To show this we pursue the calculation of $Z^{(1)}$ begun in (8.8). We there substitute the expansion

$$\mathcal{F}(x,y) = \mathcal{F}^* + \tfrac{1}{2}\mathcal{F}^*_{xx}x^2 + \tfrac{1}{2}\mathcal{F}^*_{yy}y^2 + \ldots\,. \tag{8.16}$$

We can then carry out the integrations in (8.8) by the saddle point method and find that only the quadratic terms in (8.16) contribute. The result has the form

$$Z^{(1)} \simeq \mathrm{e}^{-N\mathcal{F}^* - f_{\mathrm{int}}} \left[1 + \mathcal{O}(N^{-1})\right] \tag{8.17}$$

where

$$N\mathcal{F}^* = N\mathcal{F}(0,0) = -2N \log \lambda_+(K, H_0) \tag{8.18}$$

and

$$f_{\mathrm{int}}(K,\eta) = \tfrac{1}{2} \log \left[1 - g_0^2 \chi^2(K, H_0)\right]. \tag{8.19}$$

Here $\mathcal{F}^*$ is the free energy (divided by T) of two independent Ising chains in an effective field H_0. Since H_0 is proportional to the coupling ηK between the chains, the field dependent part of $\mathcal{F}^*$ actually represents the bulk interaction free energy $NF_{\mathrm{int}}^{(0)}$ between the chains, that is,

$$NF_{\mathrm{int}}^{(0)}(K,\eta) = \begin{cases} 0, & T \geq T_{\mathrm{c}}\,, \\ -2N \log \left(\dfrac{\lambda_+(K, H_0)}{\lambda_+(K, 0)} \right), & T < T_{\mathrm{c}}\,; \end{cases} \tag{8.20}$$

and furthermore $f_{\mathrm{int}}(K, \eta)$ is a residual interaction free energy between them which remains of order N^0 as $N \to \infty$. The energy that one drives from it has a cusp singularity and hence the exponent $\alpha = 0$ [1].

Beyond this leading order result we obtain f_{int} explicitly in terms of the two system parameters K and η by substituting in (8.19) the expressions for g_0 and χ given in (6.2c) and (8.13), respectively, and (when $T < T_{\mathrm{c}}$) eliminating H_0. The result is that

$$f_{\mathrm{int}}(K,\eta) \;=\; \begin{cases} \tfrac{1}{2} \log \left(1 - \mathrm{e}^{4K} \tanh^2 \eta K\right), & T > T_{\mathrm{c}}\,, \\ \tfrac{1}{2} \log \left(1 - \mathrm{e}^{-8K} \tanh^{-4} \eta K\right), & T < T_{\mathrm{c}}\,. \end{cases} \tag{8.21}$$

In view of (8.20) we see that F_{int} has a linear cusp at $T = T_{\mathrm{c}}$, and (8.21) shows that f_{int} diverges logarithmically for $T \to T_{\mathrm{c}}$. In spite of this weak divergence, the finite size correction f_{int} to the interaction free energy F_{int} also conforms the classical specific heat exponent $\alpha = 0$.

8.4 Finite size scaling of the free energy near T_{c}

We will show how our approach allows for finding the finite size scaling functions. By the way of an example we consider the singular part of the free energy. For $T \to T_{\mathrm{c}}$ the quantity f_{int} diverges due to the second order derivative $\mathcal{F}^*_{xx}$ becoming zero. In order for the integral (8.8) combined with (8.16) to converge at $T = T_{\mathrm{c}}$, we have to include higher order terms in the expansion (8.16). We will write

$$\mathcal{F}(x,y) = \mathcal{F}^* + \tfrac{1}{2}\mathcal{F}^*_{xx}x^2 + \tfrac{1}{2}\mathcal{F}^*_{yy}y^2 + \tfrac{1}{6}\mathcal{F}^*_{xxx}x^3 + \tfrac{1}{24}\mathcal{F}^*_{xxxx}x^4 + \ldots \tag{8.22}$$

and will argue below that near T_{c} the terms not exhibited explicitly in this series are of higher order[6]. In order to find the coefficients in (8.22 we perform a straightforward derivation of (8.9) and set $x^* = y^* = 0$. We then define

$$\epsilon = \frac{T - T_{\mathrm{c}}}{T_{\mathrm{c}}} = -\frac{K - K_{\mathrm{c}}}{K_{\mathrm{c}}} \tag{8.23}$$

which, in the vicinity of T_{c}, leads to

$$H_0 = B_{\mathrm{c}}\, \epsilon^{1/2} + \mathcal{O}(\epsilon) \tag{8.24}$$

where from (4.6) we have

$$B_{\mathrm{c}}^2 = \begin{cases} 0, & T > T_{\mathrm{c}}\,, \\ 2\mathrm{e}^{-2K_{\mathrm{c}}}\big(\eta + 1/\sinh 2K_{\mathrm{c}}\big)K_{\mathrm{c}}\,, & T < T_{\mathrm{c}}\,. \end{cases} \tag{8.25}$$

When using (8.24) in the coefficients found above we obtain

$$\begin{aligned} \mathcal{F}^*_{xx} &= a_{\pm}\epsilon + \mathcal{O}(\epsilon^2), \\ \mathcal{F}^*_{yy} &= 4\mathrm{e}^{2K_{\mathrm{c}}} + \mathcal{O}(\epsilon), \\ \mathcal{F}^*_{xxx} &= b_{\pm}(-\epsilon)^{1/2} + \mathcal{O}(\epsilon^{3/2}), \\ \mathcal{F}^*_{xxxx} &= c + \mathcal{O}(\epsilon), \end{aligned} \tag{8.26}$$

where

$$\begin{aligned} a_{\pm} &= \left\{ \begin{matrix} 2 \\ 4 \end{matrix} \right\} \big(\mathrm{e}^{4K_{\mathrm{c}}} - 1\big)\big(\eta + 1/\sinh 2K_{\mathrm{c}}\big)K_{\mathrm{c}}\,, \qquad \begin{matrix} T > T_{\mathrm{c}}\,, \\ T < T_{\mathrm{c}}\,. \end{matrix} \\ b_{\pm}^2 &= \begin{cases} 0, & T > T_{\mathrm{c}}\,, \\ 4\big(3\mathrm{e}^{4K_{\mathrm{c}}} - 1\big)\big(\eta + 1/\sinh 2K_{\mathrm{c}}\big)K_{\mathrm{c}}\,, & T < T_{\mathrm{c}}\,. \end{cases} \\ c &= 6\mathrm{e}^{2K_{\mathrm{c}}}, \end{aligned} \tag{8.27}$$

We substitute the explicit expressions (8.26) in (8.22) and use that expansion in the integral (8.8). When we introduce the scaled variables of integration u and v defined by

$$x = N^{-1/4}u, \qquad y = N^{-1/2}v, \tag{8.28}$$

as well as the scaling variable

$$\tau = \epsilon N^{1/2}, \tag{8.29}$$

[6]Terms with an odd number of y derivations vanish by symmetry.

the factor N disappears from the exponential. After carrying out the Gaussian integration on v we get

$$Z^{(1)} \simeq \mathrm{e}^{-N\mathcal{F}^*} \times \frac{N^{1/4}\mathrm{e}^{K_\mathrm{c}}}{\sqrt{2\pi}}\, \mathcal{Z}(\tau), \tag{8.30}$$

valid in the scaling limit $N \to \infty$, $T \to T_\mathrm{c}$ with τ fixed, and where $\mathcal{Z}$ is the scaling function

$$\mathcal{Z}(\tau) = \int_{-\infty}^{\infty} \mathrm{d}u \, \exp\left[-\tfrac{1}{2}a_\pm|\tau|u^2 - \tfrac{1}{6}b_\pm(-\tau)^{1/2}u^3 - \tfrac{1}{24}cu^4\right]. \tag{8.31}$$

It is of a type that occurs standardly in problems with mean field type critical behavior; they have been studied recently by Grüneberg and Hucht [8]. It has the limiting behavior

$$\mathcal{Z}(\tau) \simeq \begin{cases} \mathcal{Z}(0) \equiv \int_{-\infty}^{\infty}\mathrm{d}u\, \mathrm{e}^{-cu^4/24}, & \tau \to 0, \\ \left(\dfrac{2\pi}{a_\pm}|\tau|\right)^{1/2}, & \tau \to \pm\infty. \end{cases} \tag{8.32}$$

Upon combining (8.17) and (8.30) we find that

$$f_\mathrm{int}(K,\eta) = -\tfrac{1}{4}\log N - \log \mathcal{Z}(tN^{1/2}) - \tfrac{1}{2}\log\left(\frac{\mathrm{e}^{2K_\mathrm{c}}}{2\pi}\right) + \ldots, \tag{8.33}$$

again valid in the scaling limit, and where the dots stand for terms that vanish as $N \to \infty$. It follows, in particular, that equation (8.21) may now be completed by

$$f_\mathrm{int}(K_\mathrm{c},\eta) \simeq \tfrac{1}{4}\log N + \log \mathcal{Z}(0) + \ldots, \qquad T = T_\mathrm{c}\,, \quad N \to \infty, \tag{8.34}$$

where the dots stand for terms that vanish as $N \to \infty$.

8.5 Susceptibilities

Of primary interest are the correlations between the fluctuations of the magnetizations in the two chains. We set as before $\delta\mu = \mu - m_0$. The general expression that we will study here is

$$\begin{aligned} \chi_{k\ell} &\equiv \langle \delta\mu^k(r)\delta\mu^\ell(s)\rangle \\ &= \langle \delta\mu^k(r)\delta\mu^\ell(s)\rangle^{(1)} + \ldots \end{aligned} \tag{8.35}$$

where the dots in the last line, obtained according to (8.1), represent higher order terms. Special cases that we will consider are the cross-chain susceptibility χ_int and the single-chain susceptibility χ_sin, defined as

$$\chi_\mathrm{int} = N\chi_{11} = N\langle\delta\mu(r)\delta\mu)(s)\rangle, \tag{8.36a}$$

$$\chi_\mathrm{sin} = N\chi_{20} = N\langle\delta\mu^2(r)\rangle, \tag{8.36b}$$

in which, of course, the latter is also equal to χ_{02} by symmetry.

8.5.1 Cross-susceptibility

We first consider the correlations between the fluctuating magnetizations of the two chains. The cross-susceptibility $\chi_{\rm int}$ is the quantity most characteristic of these correlations. From equations (6.2b) and (8.3) it is clear that $\chi_{\rm int} = \partial \log Z^{(1)}/\partial g_0$ where the derivative has to be evaluated at fixed K and H_0, considering g_0 as an independent parameter in (8.4). Doing the calculation for $Z^{(1)}$ given by (8.17), (8.18), and (8.19), we observe that $\mathcal{F}^* = \mathcal{F}(0,0)$ is independent of g_0 so that

$$\begin{aligned}\chi_{\rm int}(K,\eta) = \frac{\partial f_{\rm int}}{\partial g_0} &= \frac{g_0\chi^2}{1-g_0^2\chi^2} \\ &= \begin{cases} \dfrac{\tanh \eta K}{\mathrm{e}^{-4K} - \tanh^2 \eta K} & T > T_{\rm c}\,, \\ \dfrac{\mathrm{e}^{-8K}(1-\tanh^2 \eta K)}{(\tanh \eta K)(1-\mathrm{e}^{-4K})(\tanh^4 \eta K - \mathrm{e}^{-8K})} & T < T_{\rm c}\,. \end{cases}\end{aligned} \tag{8.37}$$

For $T \to T_{\rm c}$ this quantity diverges as $|T-T_{\rm c}|^{-\gamma_{\rm int}}$ with $\gamma_{\rm int} = 1$. It is a signal that at $T = T_{\rm c}$ this correlation scales with another power of N. A scaling function for $\chi_{\rm int}$ may be derived from the one for $f_{\rm int}$, but we will not try to be exhaustive.

Since at speed $v = \infty$ all index pairs (i,j) are equivalent, the correlations between the r- and the s-spins are given by

$$\langle r_i s_j \rangle - m_0^2 = N^{-1}\chi_{\rm int}(K,\eta). \tag{8.38}$$

8.5.2 Single-chain chain susceptibility

The single-chain susceptibilities $\chi_{\rm sin}$ is defined in equation (8.36). Let us now consider the general expression (8.35) for $\chi_{k\ell}$, for which the appropriate approach differs slightly from that of the preceding subsection. One may generate insertions $\delta\mu^k(r$ [or $\delta\mu^\ell(s)$] in the integral (8.5) by passing from x and y to the two independent variables $z = (x+\mathrm{i}y)$ and $\bar{z} = (x - \mathrm{i}y)$ and letting $N^{-k}\partial^k/\partial z^k$ [or $N^{-\ell}\partial^\ell/\partial \bar{z}^\ell$] act on $\mathrm{e}^{-2zNm_0}Z(K,H_0+z)$ [or on $\mathrm{e}^{-2\bar{z}Nm_0}Z(K,H_0+\bar{z})$]. We find, using (8.6) and neglecting again the effect of λ_- which is exponentially small in N,

$$N^{-k}\frac{\partial^k}{\partial z^k}\left[\mathrm{e}^{-zNm_0}Z(K,H_0+z)\right] = J_k(z)\,Z(K,H_0+z), \tag{8.39}$$

in which

$$\begin{aligned}
J_0(z) &= 1, \\
J_1(z) &= \tilde{m} - m_0\,, \\
J_2(z) &= (\tilde{m} - m_0)^2 + N^{-1}\tilde{\chi}, \\
J_3(z) &= (\tilde{m} - m_0)^3 + 3N^{-1}(\tilde{m} - m_0)\tilde{\chi} + N^{-2}\tilde{\chi}', \\
J_4(z) &= (\tilde{m} - m_0)^4 + 6N^{-1}(\tilde{m} - m_0)^2\tilde{\chi} + 4N^{-2}(\tilde{m} - m_0)\tilde{\chi}' \\
&\quad +3N^{-2}\tilde{\chi}^2 + N^{-3}\tilde{\chi}'',
\end{aligned} \tag{8.40}$$

where, in this formula, we abbreviated $\tilde{m} = m(K, H_0+z)$ and $\tilde{\chi} = \chi(K, H_0+z)$ [see equations (4.4) and (8.13)] in order to emphasize the z dependence of these quantities, and where the primes on $\tilde{\chi}$ stand for differentiations with respect to H_0. Equations (8.39) and (8.40) of course have counterparts obtained by letting $r \mapsto s$, $k \mapsto \ell$ and $z \mapsto \bar{z}$. When (8.39) is substituted in (8.35) we obtain

$$\chi_{k\ell} = \langle J_k(z) J_\ell(\bar{z})\rangle^{(1)} + \ldots, \tag{8.41}$$

where the dots stand for higher-than-leading order terms in the N^{-1} expansion.

By virtue of equations (8.41) and (8.40) it follows that

$$\begin{aligned}
\chi_{20} &= \langle J_2(z)\rangle_{\mathrm{G}}^{(1)} \\
&= \langle \big(m(K, H_0 + z) - m_0\big)^2\rangle + N^{-1}\langle \chi(K, H_0 + z)\rangle
\end{aligned} \tag{8.42}$$

We now expand m and χ for small z anticipating that upon integration with weight $\exp(-N\mathcal{F})$ each factor z^2 will, to leading order, produce a factor N^{-1}. After multiplication by N this yields

$$\chi_{\sin}(K, \eta) = N\chi^2(K, H_0)\langle z^2\rangle_{\mathrm{G}}^{(1)} + \chi(K, H_0) + \mathcal{O}(N^{-1}). \tag{8.43}$$

Anticipating again that each factor z or $\bar{z}$ will produce a factor $N^{-1/2}$, we see that all terms exhibited explicitly on the right hand sides in (8.48) are of order N^{-1}. We have replaced the averages $\langle \ldots \rangle^{(1)}$, which are with respect to $\exp(-N\mathcal{F}(x, y)$, by averages $\langle \ldots \rangle_{\mathrm{G}}^{(1)}$ in which $\mathcal{F}(x, y)$ of equation (8.9) is replaced with the Gaussian terms in its expansion, shown in (8.16).

Upon using in (8.42) the explicit evaluations

$$\begin{aligned}
\langle z^2\rangle_{\mathrm{G}}^{(1)} &= \langle x^2\rangle_{\mathrm{G}}^{(1)} - \langle y^2\rangle_{\mathrm{G}}^{(1)} \\
&= \frac{1}{N}\left(\frac{1}{\mathcal{F}^*_{xx}} - \frac{1}{\mathcal{F}^*_{yy}}\right) = \frac{g_0^2\chi}{N(1 - g_0^2\chi^2)}\,, \qquad T \neq T_{\mathrm{c}}\,.
\end{aligned} \tag{8.44}$$

we arrive at

$$\chi_{\rm sin}(K,\eta) = \frac{\chi}{1-g_0^2\chi^2}\,, \qquad T \neq T_{\rm c}\,, \tag{8.45}$$

valid in the limit $N \to \infty$. Hence the in-chain susceptibility $\chi_{\rm sin}$ is equal to the susceptibility of the 1D Ising model enhanced by a factor $(1-g_0^2\chi^2)^{-1}$ due to the presence of the other chain.

Using expressions (6.2c) and (8.13) for g_0 and χ, respectively, we may render (8.45) explicit in terms of K and η and get

$$\chi_{\rm sin}(K,\eta) = \begin{cases} \dfrac{\mathrm{e}^{-2K}}{\mathrm{e}^{-4K} - \tanh^2\eta K}\,, & T > T_{\rm c}\,, \\[2ex] \dfrac{\mathrm{e}^{-4K}(\tanh\eta K)(1-\tanh^2\eta K)}{(1-\mathrm{e}^{-4K})(\tanh^4\eta K - \mathrm{e}^{-8K})}\,, & T < T_{\rm c}\,. \end{cases} \tag{8.46}$$

For $T \to T_{\rm c}$ the susceptibility $\chi_{\rm sin}$ diverge as $(T-T_{\rm c})^{-\gamma}$ with, again, the classical critical exponent $\gamma = 1$. For $\eta = 0$ (whence $T_{\rm c} = 0$) the first one of equations (8.46) reduces to the standard susceptibility of the zero field 1D Ising chain.

In agreement with the symmetry of the problem, $\chi_{\rm int}$ is odd and $\chi_{\rm sin}$ is even in η. Both above and below $T_{\rm c}$ one easily verifies that in agreement with Schwarz's inequality we have $\chi_{\rm int}/\chi_{\rm sin} \leq 1$.

8.6 Spontaneous magnetization

For $T \geq T_{\rm c}$ symmetry dictates that the magnetization $\langle\mu(r)\rangle$ and $\langle\mu(s)\rangle$ are zero to all orders. However, for $T < T_{\rm c}$ the magnetization $\mu(r) = N^{-1}\sum_{j=1}^N r_j$ has, to leading order, a Gaussian probability distribution of width $N^{-1/2}$ around $m_0(K,H_0)$. As a consequence $\langle\delta\mu(r)\rangle$ vanishes to order $N^{-1/2}$. However, to order N^{-1} there appear nonzero corrections terms to $\langle\mu(r)\rangle$. As an application of equation (8.1) we calculate in this subsection these correction terms.

Upon using (8.1) for the spacial case $A = \delta\mu(r)$ and inserting in it the explicit expression (7.27) for q_1 we obtain

$$\langle\delta\mu(r)\rangle = \langle\delta\mu(r)\rangle^{(1)} + \tfrac{2}{3}Ng_0^2\tanh H_0\left[\langle J_4(z)\rangle^{(1)} + \langle J_1(z)J_3(\bar z)\rangle^{(1)}\right]. \tag{8.47}$$

When substituting (8.40) in the second term of (8.47) we see that we need

$$\begin{aligned} \langle J_4(z)\rangle^{(1)} &= \chi^4\langle z^4\rangle_{\rm G}^{(1)} + 6N^{-1}\chi^3\langle z^2\rangle_{\rm G}^{(1)} + 3N^{-2}\chi^2 + \mathcal{O}(N^{-5/2}), \\ \langle J_1(z)J_3(\bar z)\rangle^{(1)} &= \chi^4\langle z\bar z^3\rangle_{\rm G}^{(1)} + 3N^{-1}\chi^3\langle z\bar z\rangle_{\rm G}^{(1)} + \mathcal{O}(N^{-5/2}). \end{aligned} \tag{8.48}$$

We have replaced the averages $\langle\ldots\rangle^{(1)}$ by averages $\langle\ldots\rangle_{\rm G}^{(1)}$ for the same reasons as in the preceding subsection. Taking into account again that each factor

$z$or $\bar{z}$ brings in a power $N^{-1/2}$, we see that all terms explicitly exhibited on the right hand sides of equations (8.48) are of the same order in N, namely $\mathcal{O}(N^{-2})$. The Gaussian averages are easily calculated and we are led to

$$\langle J_4(z)\rangle^{(1)} \,+\, \langle J_1(z)J_3(\bar{z})\rangle^{(1)} = \frac{3\chi^3(\chi+g_0)}{N^2(1-g_0^2\chi^2)^2} + \mathcal{O}(N^{-5/2}). \tag{8.49}$$

We should now evaluate the first term on the right hand side of (8.47), namely

$$\langle\delta\mu(r)\rangle^{(1)} = \langle J_1(z)\rangle^{(1)} = \chi\langle z\rangle^{(1)}. \tag{8.50}$$

The Gaussian average $\langle z\rangle_{\mathrm{G}}^{(1)}$ vanishes on account of symmetry. However, when the third order terms in the Taylor expansion (8.16) of $\mathcal{F}(x,y)$ are kept and we expand these we get after a straightforward calculation that we will not reproduce here,

$$\begin{aligned}
\langle\delta\mu(r)\rangle^{(1)} &= \tfrac{1}{3}N\chi\chi'\left[\langle x^4\rangle_{\mathrm{G}}^{(1)} - 3\langle x^2y^2\rangle_{\mathrm{G}}^{(1)}\right] \\
&= N\chi\chi'\langle x^2\rangle_{\mathrm{G}}^{(1)}\left[\langle x^2\rangle_{\mathrm{G}}^{(1)} - \langle y^2\rangle_{\mathrm{G}}^{(1)}\right] + \mathcal{O}(N^{-2}) \\
&= \frac{g_0^3\chi^2\chi'}{2N(1-g_0\chi)^2(1+g_0\chi)} + \mathcal{O}(N^{-2}).
\end{aligned} \tag{8.51}$$

The final result for $\langle\delta\mu(r)\rangle$ is obtained by substitution of (8.51) and (8.49) in (8.47). We see that $\langle\delta\mu(r)\rangle$ has two contributions of order N^{-1}. The contribution $\langle\delta\mu(r)\rangle^{(1)}$ comes from the effective leading order Hamiltonian $\mathcal{H}^{(1)}$. The second contribution accompanies the violation of detailed balancing symmetry and is therefore essentially a non-thermodynamic effect.

8.7 Pair correlation function

It is of interest to study the pair correlation

$$g_N(\ell) \equiv \langle r_j r_{j+\ell}\rangle \tag{8.52}$$

in a single chain. To that end we consider again expansion (8.1), now with $A = r_j r_{j+\ell}$. Its first term may be written

$$g_N^{(1)}(\ell) = Z_\ell^{(1)}/Z^{(1)} \tag{8.53}$$

where $Z_\ell^{(1)}$ is given by (8.4) but with an insertion $r_j r_{j+\ell}$ in the sum on r. Equivalently, $Z^{(1)}$ is given by the same integral as (8.8) but with an insertion $\tilde{g}_N^{(1)}(\ell;K,H_0+z)$, this quantity being the pair correlation of the 1D Ising chain in a field $H_0+x+\mathrm{i}y$. Evaluation by means of the standard transfer matrix method yields

$$\tilde{g}_N^{(1)}(\ell;K,H_0+z) = m^2(K,H_0+z) + \frac{\mathrm{e}^{-4K}\,\tilde{\Lambda}^\ell(K,H_0+z)}{\sinh^2(H_0+z)+\mathrm{e}^{-4K}}\,, \tag{8.54}$$

well-known in the case $z = 0$, in which we defined $\tilde{\Lambda} = \lambda_-/\lambda_+$, where the tilde serves as a reminder of the z dependence, and where contributions exponentially small in N have again been neglected. In order to obtain the desired physical correlation function $g_N(\ell)$ of this system we now have to average (8.54) with an appropriately normalized weight $\exp\big[-N\mathcal{F}(x,y)\big]$.

We will consider this quantity in the high-temperature regime $T > T_{\rm c}$ where $H_0 = 0$. Knowing that z is of order $N^{-1/2}$ we expand (8.54) for small z, which gives

$$\begin{aligned} \tilde{g}_N^{(1)}(\ell; K, H_0 + z) \quad &= \\ &\mathrm{e}^{4K} z^2 \quad + \quad (\tanh^\ell K)(1 - \mathrm{e}^{4K} z^2) \exp\big(-(\mathrm{e}^{-4K} + \mathrm{e}^{2K}\ell) z^2\big) + \mathcal{O}(N^{-2}) \end{aligned} \tag{8.55, 8.56}$$

To leading order the average on z may be carried out with the weight $\exp\big[-N\mathcal{F}(x,y)\big]$ in which the expansion $\mathcal{F}$ is limited to its quadratic terms. Straightforward calculation yields

$$g_N(\ell) = \tanh^\ell K + (1 - \tanh^\ell K) \frac{g_0^2 \chi^3}{1 - g_0^2 \chi^2} N^{-1} + \mathcal{O}(N^{-2}), \qquad T > T_{\rm c}, \tag{8.57}$$

valid for $N \to \infty$ at fixed ℓ, where as before χ stands for the susceptibility $\chi(K, 0) = \mathrm{e}^{2K}$ of the 1D Ising chain and where $g_0 = \tanh \eta K$. In the scaling limit $\ell, N \to \infty$ with a fixed ratio one obtains

$$g_N(\ell) \simeq (\tanh^\ell K) \phi(\ell N^{-1}) + \frac{g_0^2 \chi^3}{1 - g_0^2 \chi^2} N^{-1}, \quad T > T_{\rm c}, \quad \ell, N \to \infty, \tag{8.58}$$

in which each of the two terms is valid up to corrections of relative order N^{-1} and in which ϕ is the scaling function defined by

$$\phi^2(x) = \frac{1 - g_0^2 \chi^2}{1 - g_0^2 \chi^2 + 2 g_0^2 \chi^2 x}. \tag{8.59}$$

We observe the noncommutativity

$$\lim_{N\to\infty} \sum_{\ell=-N/2+1}^{N/2} g_N(\ell) \neq \sum_{\ell=-\infty}^{\infty} \lim_{N\to\infty} g_N(\ell). \tag{8.60}$$

The right hand side of this inequality is equal to $\chi(K, 0)$ whereas the right hand side is equal to $\chi(K, 0) + \chi_{\rm int}(K, \eta)$.

We conclude by noting that the pair correlation function may also be studied to higher order in N^{-1} in the low-temperature regime. For $T < T_{\rm c}$ the fluctuations of the magnetic field z are asymmetric and greater care is required. We will not include such a calculation here.

9 Traffic model

Motivated by an interest very different from that of references [1, 2] we recently introduced a new traffic model describing vehicles that may overtake each other on a road with two opposite lanes [9]. That work shows the appearance of a phase transition when the traffic intensity, supposed equal on the two lanes, attains a critical value. Above the critical intensity the symmetry between the two traffic lanes is broken: one lane has dense and slow, the other one dilute and fast traffic. The study of reference [9] invoked a mean-field-type assumption that couples the velocity of a vehicle in a given lane to the *average* of the vehicle velocities in the opposite lane. This assumption was justified by the argument that a vehicle in one lane encounters, in the course of time, all vehicles in the opposite lane. Although there is no one-to-one correspondence between the two models, they share essentially the same features, as may be seen as follows. For $J_2 < 0$ the two chains of the CRIC studied here have opposite spontaneous magnetizations; up-spins may then be regarded as the vehicles of the traffic problem; they will be denser in one chain (traffic lane) than in the other. The CRIC is more amenable to analysis than the traffic model. It was shown analytically [1, 2] that the CRIC phase transition disappears when v is finite. Our simulations [10] of the traffic model have shown, nevertheless, that this problem is close to the critical point $v = \infty$. This explains the critical-point-like phenomena that we observed, namely fluctuations that last longer than the simulation time.

10 Conclusion

We have considered in this paper the nonequilibrium steady state (NESS) of a model consisting of two counter-rotating interacting Ising chains introduced by Kadau *et al.* [2] and by Hucht [1]. The model is related to a road traffic model studied earlier by ourselves [9]. Its dynamics is governed by a master equation parametrized by two interaction constants J/T and η. The model has a phase transition, known to be of mean field type, at a critical temperature $T = T_{\rm c}$.

Starting from the master equation we have shown that in the limiting case of a relative velocity $v = \infty$ of the two chains, the stationary state distribution $P_{\rm st}$ may be studied in an expansion in powers of the inverse system size N^{-1}. Knowing this distribution we have calculated, also as expansions in N^{-1}, of averages of physical interest: the interaction free energy between the chains, the in-chain and cross-chain susceptibilities, the correlation function (for $T > T_{\rm c}$), and the spontaneous magnetization (for $T < T_{\rm c}$). We have shown how near criticality scaling functions may be explicitly calculated.

Whereas to leading order the force exerted by one chain on the other is that of an effective magnetic field H_0, the N^{-1} expansion requires that we

take into account the fluctuations of this field around its average. It then appears that to leading order the dynamics obeys detailed balancing with respect to an effective Hamiltonian, as was found by Hucht [1], but that to higher order in the expansion the detailed balancing is violated.

In this work we have addressed many different, albeit interrelated, aspects of the finite-size CRIC. We have not tried to be exhaustive and have not considered, for example, energy dissipation. Similarly, the parallel problem with open boundary conditions has been left aside. We hope that the results of this work will be helpful in guiding the study, which we believe to be worthwhile, of the finite-velocity ($v < \infty$) version of the model.

Acknowledgments

The author thanks Cécile Appert-Rolland and Grégory Schehr (Orsay, France) and Samyr Jácobe (UFRN, Natal, Brazil) for discussions on and around the subject of this work.

References

[1] A. Hucht, *Phys. Rev. E* **80**, 061138 (2009).

[2] D. Kadau, A. Hucht, and D.E. Wolf, *Phys. Rev. Lett.* **101**, 137205 (2008).

[3] R.J. Glauber, *J. Math. Phys.* **4**, 294 (1963).

[4] V. Démery and D.S. Dean, *Phys. Rev. Lett.* **104**, 080601 (2010).

[5] M. Pleimling, B. Schmittmann, and R.K.P. Zia, *Europhys. Lett.* **89**, 50001 (2010).

[6] A. Prados, L.L. Bonilla, and A. Carpio, *J. Stat. Mech.* P06016 (2010).

[7] F. Iglói, M. Pleimling, and L. Turban, `arXiv:1010.0738 cond-mat.stat.-mech`

[8] D. Grüneberg and A. Hucht, *Phys. Rev. E* **69**,036104 (2004).

[9] C. Appert-Rolland, H.J. Hilhorst, and G. Schehr, *J. Stat. Mech.* P08024 (2010).

[10] C. Appert-Rolland, H.J. Hilhorst, and G. Schehr, *unpublished.*